

## MOTION OF LEVEL SETS BY MEAN CURVATURE. II

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**ABSTRACT.** We give a new proof of short time existence for the classical motion by mean curvature of a smooth hypersurface. Our method consists in studying a fully nonlinear uniformly parabolic equation satisfied by the signed distance function to the surface

### 1. INTRODUCTION

We present in this paper a new, elementary, and fairly concise proof of short time existence for the classical motion of a smooth hypersurface evolving according to its mean curvature. In this problem we are given initially a smooth connected hypersurface  $\Gamma_0$ , which is the boundary of a bounded open set  $U \subset \mathbf{R}^n$ . We then allow  $\Gamma_0$  to evolve in time into a family of surfaces  $\{\Gamma_t\}_{t>0}$  by moving each point on  $\Gamma_t$  ( $t \geq 0$ ) at a velocity equal to  $(n-1)$  times the mean curvature vector at that point. Our intent is to verify that for small times at least, the classical motion as thus envisioned in fact exists and is unique. This assertion was first proved by Gage and Hamilton [4], based on the work of Hamilton [5], and we discuss below the relation of our work to theirs.

This is a companion to our paper [3], wherein we defined and then studied a generalized notion of evolution via mean curvature, existing for all times and agreeing with the classical motion, if and so long as the latter exists. This generalized evolution is constructed in [3] by first building an appropriately defined unique weak solution of the PDE

$$(1.1) \quad u_t = \left( \delta_{ij} - \frac{u_{x_i} u_{x_j}}{|Du|^2} \right) u_{x_i x_j} \quad \text{in } \mathbf{R}^n \times (0, \infty),$$

$$(1.2) \quad u = g \quad \text{on } \mathbf{R}^n \times \{t = 0\}.$$

Equation (1.1) says that *each* level set of  $u$  evolves according to its mean curvature, at least in regions where  $u$  is smooth and  $|Du| \neq 0$ . Given then  $\Gamma_0$  as above, we select a smooth function  $g: \mathbf{R}^n \rightarrow \mathbf{R}$  for which

$$(1.3) \quad \Gamma_0 = \{x \in \mathbf{R}^n \mid g(x) = 0\}.$$

We next write

$$(1.4) \quad \Gamma_t \equiv \{x \in \mathbf{R}^n \mid u(x, t) = 0\} \quad (t \geq 0),$$

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and call the family of sets  $\{\Gamma_t\}_{t \geq 0}$  so defined the *generalized evolutions* by mean curvature starting from  $\Gamma_0$ . The rest of [3] is devoted to deducing from the PDE (1.1), (1.2) various elementary geometric properties of the flow  $\Gamma_0 \mapsto \Gamma_t$  ( $t \geq 0$ ). A very similar investigation of this and more complicated problems has been independently developed by Chen, Giga, and Goto [2].

In this paper we return again to the idea of studying a nonlinear PDE, a level set of whose solution evolves via mean curvature. Our idea is first to assume that  $\Gamma_0$  develops by classical mean curvature motion, at least for times  $0 \leq t \leq t_0$ , and then to derive the PDE verified by  $d$ , the signed distance function to the surface at each time. This turns out to be a fully nonlinear, uniformly parabolic equation: see (1.9) below. Next we construct for short times a smooth solution of this equation, subject to nonlinear boundary conditions in an appropriate region, and then finally verify that our solution is in fact the signed distance function to a family of smooth surfaces evolving from  $\Gamma_0$  by mean curvature motion.

We proceed now to the heuristic derivation of our PDE. Suppose therefore we are given the smooth hypersurface  $\Gamma_0 = \partial U$  as above, a time  $t_0 > 0$ , and a classical evolution  $\{\Gamma_t\}_{0 \leq t \leq t_0}$  of surfaces developing from  $\Gamma_0$  by mean curvature flow. Then for each time  $t$  in  $[0, t_0]$ ,  $\Gamma_t$  is the smooth boundary of a bounded open set  $U_t$ , which is diffeomorphic to  $U = U_0$ . Define the *signed distance function*

$$(1.5) \quad d(x, t) \equiv \begin{cases} \text{dist}(x, \Gamma_t) & (x \in \mathbf{R}^n - \overline{U}_t), \\ -\text{dist}(x, \Gamma_t) & (x \in U_t) \end{cases}$$

for  $x \in \mathbf{R}^n$ ,  $0 \leq t \leq t_0$ . By assumption  $\Gamma \equiv \bigcup_{0 \leq t \leq t_0} \Gamma_t \times \{t\}$  is smooth, and thus  $d$  is smooth in

$$Q^+ \equiv \{(x, t) \mid 0 < t < t_0, 0 < d(x, t) < \delta_0\}$$

and in

$$Q^- = \{(x, t) \mid 0 < t < t_0, -\delta_0 < d(x, t) < 0\}$$

for  $\delta_0 > 0$  small enough. Fix any point  $(x, t) \in Q^+$ . Then provided  $\delta_0 > 0$  is sufficiently small, there exists a unique point  $y \in \Gamma_t$  for which  $d(x, t) = |x - y|$ . Let  $\nu = Dd$  be the smooth unit normal vector field pointing from  $\Gamma$  into  $Q^+$ . As  $\{\Gamma_t\}_{0 \leq t \leq t_0}$  moves via mean curvature, we have

$$d_t(x, t) = \text{div}(\nu)(y, t).$$

On the other hand the (unordered) eigenvalues of the matrix  $D^2d(x, t)$  are

$$(1.6) \quad \begin{aligned} \lambda_i &= \lambda_i(D^2d(x, t)) = -\kappa_i / (1 - \kappa_i d(x, t)) \quad (1 \leq i \leq n-1) \\ \lambda_n &= \lambda_n(D^2d(x, t)) = 0, \end{aligned}$$

$\{\kappa_1, \dots, \kappa_{n-1}\}$  denoting the principal curvatures of  $\Gamma_t$  at  $y$ , computed with respect to the normal field  $\nu$ . Inverting (1.6) we compute

$$\kappa_i = \frac{\lambda_i}{\lambda_i d(x, t) - 1} \quad (1 \leq i \leq n-1).$$

Since

$$\text{div}(\nu) = -(\kappa_1 + \dots + \kappa_{n-1}),$$

we deduce finally

$$(1.7) \quad d_t(x, t) = f(\lambda_1(D^2 d(x, t)), \dots, \lambda_n(D^2 d(x, t)), d(x, t))$$

for

$$(1.8) \quad f(\lambda_1, \dots, \lambda_n, z) \equiv \sum_{i=1}^n \frac{\lambda_i}{1 - \lambda_i z}.$$

The same formula results if  $(x, t) \in Q^-$ .

Now our PDE (1.7) has the general form

$$(1.9) \quad d_t = F(D^2 d, d);$$

and, as  $f$  is symmetric in the variables  $\lambda_1, \dots, \lambda_n$ ,  $F$  is smooth. Since

$$(1.10) \quad \frac{\partial f}{\partial \lambda_i} = \frac{1}{(1 - \lambda_i d)^2} > 0 \quad (1 \leq i \leq n),$$

equation (1.9) is uniformly parabolic. See Caffarelli, Nirenberg, and Spruck [1] for more details.

Our plan is to study directly the PDE (1.7), (1.8). For this suppose now  $\Gamma_0$  is the smooth connected boundary of a bounded open set  $U \subset \mathbf{R}^n$ , and let

$$(1.11) \quad g(x) \equiv \begin{cases} \text{dist}(x, \Gamma_0) & (x \in \mathbf{R}^n - \bar{U}), \\ -\text{dist}(x, \Gamma_0) & (x \in U) \end{cases}$$

be the signed distance function. Fix then  $\delta_0 > 0$  so small that  $g$  is smooth within

$$(1.12) \quad V \equiv \{x \in \mathbf{R}^n \mid -\delta_0 < g < \delta_0\},$$

and write

$$Q \equiv V \times (0, t_0), \quad \Sigma \equiv \partial V \times [0, t_0].$$

In §2 we construct a smooth solution  $v$  to the PDE

$$(1.13) \quad \begin{cases} v_t = F(D^2 v, v) & \text{in } Q, \\ |Dv|^2 = 1 & \text{on } \Sigma, \\ v = g & \text{on } V \times \{t = 0\} \end{cases}$$

for some small time  $t_0 > 0$ , our argument being in effect a special case of the Inverse Function Theorem. We then verify in §3 that

$$(1.14) \quad |Dv|^2 = 1 \quad \text{in } Q.$$

Setting

$$\Gamma_t \equiv \{x \in V \mid v(x, t) = 0\} \quad (0 \leq t \leq t_0),$$

we deduce from (1.13) and (1.14) that  $\{\Gamma_t\}_{0 \leq t \leq t_0}$  is a smooth evolution governed by mean curvature and  $v = d$  is the corresponding signed distance function.

Section 4 utilizes the PDE (1.7), (1.8) to prove an “instantaneous tearing apart” assertion for close-by evolving surfaces. Some consequences are discussed in [3, §8].

As noted, Gage and Hamilton in [4, §2] have previously established the short-time existence of the classical mean curvature evolution by studying the degenerate parabolic system describing the parametrized surface. As this system is

degenerately parabolic, they must appeal to a general theorem of Hamilton [5, Theorem 5.1] which employs fairly complicated techniques related to the Nash-Moser Implicit Function Theorem. Our methods are simpler.

## 2. SOLVING THE NONLINEAR PDE

Our goal in this section is to construct a smooth solution of the PDE (1.13) for  $\delta_0 > 0$  and  $t_0 > 0$  small enough. Let us first of all select  $\delta_0$  so small that

$$(2.1) \quad M\delta_0 \leq \frac{1}{4}$$

for

$$(2.2) \quad M \equiv \max_V |D^2 g|.$$

Set

$$G \equiv \{R \in S^{n \times n}, z \in \mathbf{R} \mid |z| < \delta_0, |R| < 2M\}.$$

Then since (2.1) implies

$$|\lambda_i(R)z| \leq |R||z| \leq \frac{1}{2} \quad (1 \leq i \leq n)$$

if  $(R, z) \in G$ ,

$$(2.3) \quad F(R, z) = f(\lambda_1(R), \dots, \lambda_n(R), z) = \sum_{i=1}^n \frac{\lambda_i(R)}{1 - \lambda_i(R)z}$$

is defined and smooth on  $G$ . Arbitrarily extend  $F$  off  $G$  to be smooth on all of  $S^{n \times n} \times \mathbf{R}$ , with  $|F|$ ,  $|DF|$ ,  $|D^2 F|$  bounded.

We now check that our PDE is uniformly parabolic near  $g$ .

**Lemma 2.1.** *Then exists a constant  $\theta > 0$  so that*

$$(2.4) \quad \frac{\partial F}{\partial r_{ij}}(R, z)\xi_i\xi_j \geq \theta|\xi|^2 \quad (\xi \in \mathbf{R}^n)$$

in  $V$  for each  $(R, z) \in G$ .

*Proof.* Fix  $\xi \in \mathbf{R}^n$  and choose then  $t > 0$  so small that  $(R + t\xi \otimes \xi, z) \in G$ . Using Courant's minimax characterization we deduce that the eigenvalues can be ordered so that

$$\lambda_k(R + t\xi \otimes \xi) \geq \lambda_k(R) \quad (k = 1, \dots, n).$$

Consequently

$$\begin{aligned} & F(R + t\xi \otimes \xi, z) - F(R, z) \\ &= f(\dots, \lambda_k(R + t\xi \otimes \xi), \dots, z) - f(\dots, \lambda_k(R), \dots, z) \\ &= \sum_{i=1}^n \int_0^1 \frac{\partial f}{\partial \lambda_k}(\dots, s\lambda_k(R + t\xi \otimes \xi) \\ &\quad + (1-s)\lambda_k(R), \dots, z) ds [\lambda_k(R + t\xi \otimes \xi) - \lambda_k(R)] \\ &\geq \theta \text{trace}(R + t\xi \otimes \xi - R) \end{aligned}$$

for some  $\theta > 0$  by (1.10). Dividing by  $t > 0$  and sending  $t \rightarrow 0^+$ , we arrive at inequality (2.4).  $\square$

We will seek our solution  $v$  of the PDE (1.13) in the form

$$(2.5) \quad v = g + th + w,$$

where

$$(2.6) \quad h \equiv F(D^2 g, g).$$

Then plugging into (1.13) we compute

$$w_t - a_{ij} w_{x_i x_j} + c w = A(D^2 w, w, x, t) \quad \text{in } Q$$

for

$$(2.7) \quad a_{ij} \equiv \frac{\partial F}{\partial r_{ij}}(D^2 g, g), \quad c \equiv \frac{-\partial F}{\partial z}(D^2 g, g)$$

and

$$(2.8) \quad \begin{aligned} A(R, z, x, t) &\equiv F(D^2 g + t D^2 h + R, g + t h + z) \\ &- F(D^2 g, g) - \frac{\partial F}{\partial r_{ij}}(D^2 g, g) r_{ij} - \frac{\partial F}{\partial z}(D^2 g, g) z. \end{aligned}$$

We next insert (2.5) into the nonlinear boundary condition from the PDE (1.13). Recalling that  $|Dg|^2 = 1$  and  $Dg$  is normal to  $\partial V$  we discover

$$\frac{\partial w}{\partial \nu} = a(Dw, x, t) \quad \text{on } \Sigma,$$

where  $\nu$  is the outer unit normal vector field along  $\Sigma$  and

$$(2.9) \quad a(p, x, t) \equiv \begin{cases} -\frac{1}{2}|tDh + p|^2 - t\frac{\partial h}{\partial \nu} & \text{on } \{d = \delta_0\}, \\ \frac{1}{2}|tDh + p|^2 - t\frac{\partial h}{\partial \nu} & \text{on } \{d = -\delta_0\}. \end{cases}$$

Combining everything above, we hereafter seek a smooth function  $w$  satisfying

$$(2.10) \quad \begin{cases} w_t - a_{ij} w_{x_i x_j} + c w = A(D^2 w, w, x, t) & \text{in } Q, \\ \frac{\partial w}{\partial \nu} = a(Dw, x, t) & \text{on } \Sigma, \\ w = 0 & \text{on } V \times \{t = 0\}. \end{cases}$$

We intend to solve (2.10) by finding a fixed point of the mapping  $T$  defined by inserting a given function into the nonlinear terms  $A$ ,  $a$  and solving the resulting linear PDE.

We will work in certain standard parabolic-type Hölder spaces. Fix  $0 < \alpha < 1$ . We define several norms, following Ladyzhenskaja, Solonnikov, and Ural'tseva [6, pp. 7–8]. Set

$$\begin{aligned} |u|^{(0)} &\equiv \sup\{|u(x, t)| \mid (x, t) \in Q\}, \\ |u|^{(1)} &\equiv |u|^{(0)} + |Du|^{(0)}, \\ |u|^{(2)} &\equiv |u|^{(1)} + |D^2 u|^{(0)} + |u_t|^{(0)}, \\ \langle u \rangle_x^{(\beta)} &\equiv \sup \left\{ \frac{|u(x, t) - u(y, t)|}{|x - y|^\beta} \mid (x, t), (y, t) \in Q, x \neq y \right\}, \\ \langle u \rangle_t^{(\beta)} &\equiv \sup \left\{ \frac{|u(x, t) - u(x, s)|}{|t - s|^\beta} \mid (x, t), (x, s) \in Q, t \neq s \right\}, \end{aligned}$$

for  $0 < \beta < 1$ . Then

$$\begin{aligned} \|u\|_{C^{\alpha, \frac{\alpha}{2}}(\overline{Q})} &\equiv |u|^{(0)} + \langle u \rangle_x^{(\alpha)} + \langle u \rangle_t^{(\frac{\alpha}{2})}, \\ \|u\|_{C^{1+\alpha, \frac{1+\alpha}{2}}(\overline{Q})} &\equiv |u|^{(1)} + \langle Du \rangle_x^{(\alpha)} + \langle u \rangle_t^{(\frac{1+\alpha}{2})} + \langle Du \rangle_t^{(\frac{\alpha}{2})}, \\ \|u\|_{C^{2+\alpha, \frac{2+\alpha}{2}}(\overline{Q})} &\equiv |u|^{(2)} + \langle Du \rangle_t^{(\frac{1+\alpha}{2})} + \langle D^2 u \rangle_x^{(\alpha)} + \langle u_t \rangle_x^{(\alpha)} + \langle D^2 u \rangle_t^{(\frac{\alpha}{2})} + \langle u_t \rangle_t^{(\frac{\alpha}{2})}. \end{aligned}$$

Note

$$\|Du\|_{C^{1+\alpha, \frac{1+\alpha}{2}}(\bar{Q})}, \|D^2u\|_{C^{\alpha, \frac{\alpha}{2}}(\bar{Q})}, \|u_t\|_{C^{\alpha, \frac{\alpha}{2}}(\bar{Q})} \leq \|u\|_{C^{2+\alpha, \frac{2+\alpha}{2}}(\bar{Q})}.$$

Finally define

$$\|u\|_{C^{1+\alpha, \frac{1+\alpha}{2}}(\Sigma)} \equiv \inf \left\{ \|v\|_{C^{1+\alpha, \frac{1+\alpha}{2}}(\bar{Q})} \mid v = u \text{ on } \Sigma \right\}.$$

We consider now the linear, uniformly parabolic PDE

$$(2.11) \quad \begin{cases} w_t - a_{ij}w_{x_i x_j} + cw = B & \text{in } Q, \\ \frac{\partial w}{\partial \nu} = b & \text{on } \Sigma, \\ w = 0 & \text{on } V \times \{t = 0\}. \end{cases}$$

Suppose as well

$$(2.12) \quad b = 0 \quad \text{on } \partial V \times \{t = 0\}.$$

**Lemma 2.2.** Assume  $B \in C^{\alpha, \frac{\alpha}{2}}(\bar{Q})$ ,  $b \in C^{1+\alpha, \frac{1+\alpha}{2}}(\Sigma)$ , and the zeroth order compatibility condition (2.12) holds. Then there exists a unique solution  $w \in C^{2+\alpha, \frac{2+\alpha}{2}}(\bar{Q})$  of the PDE (2.11), with the estimate

$$(2.13) \quad \|w\|_{C^{2+\alpha, \frac{2+\alpha}{2}}(\bar{Q})} \leq C(\|B\|_{C^{\alpha, \frac{\alpha}{2}}(\bar{Q})} + \|b\|_{C^{1+\alpha, \frac{1+\alpha}{2}}(\Sigma)}).$$

The constant  $C$  does not depend on  $t_0$ .

*Proof.* Except for the last assertion, this is Theorem IV.5.3 in Ladyzhenskaja, Solonnikov, Ural'tseva [6, pp. 320–321]. To obtain the statement that the constant  $C$  remains bounded even for small  $0 < t_0 < 1$ , we define

$$\begin{aligned} \tilde{Q} &= V \times (0, 1), \quad \tilde{\Sigma} = \partial V \times (0, 1), \\ \tilde{B}(x, t) &= \begin{cases} B(x, t) & \text{if } 0 \leq t \leq t_0, \\ B(x, t_0) & \text{if } t_0 \leq t \leq 1, \end{cases} \\ \tilde{b}(x, t) &= \begin{cases} b(x, t) & \text{if } 0 \leq t \leq t_0, \\ b(x, t_0) & \text{if } t_0 \leq t \leq 1. \end{cases} \end{aligned}$$

Then

$$\|\tilde{B}\|_{C^{\alpha, \frac{\alpha}{2}}(\tilde{Q})} = \|B\|_{C^{\alpha, \frac{\alpha}{2}}(\bar{Q})}, \quad \|\tilde{b}\|_{C^{1+\alpha, \frac{1+\alpha}{2}}(\tilde{\Sigma})} = \|b\|_{C^{1+\alpha, \frac{1+\alpha}{2}}(\Sigma)}.$$

Let  $\tilde{w} \in C^{2+\alpha, \frac{2+\alpha}{2}}(\tilde{Q})$  solve the PDE (2.11) with  $\tilde{Q}$ ,  $\tilde{\Sigma}$ ,  $\tilde{B}$ ,  $\tilde{b}$  replacing  $Q$ ,  $\Sigma$ ,  $B$ ,  $b$ . The estimate (2.13) then is valid for  $\tilde{w}$  with  $\tilde{Q}$ ,  $\tilde{\Sigma}$  replacing  $Q$ ,  $\Sigma$ . But  $\tilde{w} = w$  on  $Q$ , by uniqueness for parabolic equations.  $\square$

We will henceforth work in the Banach space

$$X \equiv \{w \in C^{2+\alpha, \frac{2+\alpha}{2}}(\bar{Q}) \mid w = 0 \text{ on } V \times \{t = 0\}\}.$$

Given  $\hat{w} \in X$ , we set

$$(2.14) \quad \begin{cases} B(x, t) \equiv A(D^2\hat{w}, \hat{w}, x, t), \\ b(x, t) \equiv a(D\hat{w}, x, t) \end{cases}$$

for  $A$ ,  $a$  defined by (2.8), (2.9). Now write  $T(\hat{w}) = w$ , when  $w \in X$  solves the linear PDE (2.11), for  $B$ ,  $b$  as in (2.14). We seek a fixed point of the mapping  $T: X \rightarrow X$ . Give  $r_0 > 0$ , set

$$Y \equiv \{w \in X \mid \|w\|_{C^{2+\alpha, \frac{2+\alpha}{2}}(\bar{Q})} \leq r_0\}.$$

**Lemma 2.3.** *If  $t_0, r_0 > 0$  are sufficiently small, then*

$$(2.15) \quad T: Y \rightarrow Y.$$

*Proof.* 1. Choose any function  $\hat{w} \in Y$ , define  $B, b$  by (2.14), and let  $w \in X$  solve (2.11). We must show the inequality

$$(2.16) \quad \|\hat{w}\|_{C^{2+\alpha, \frac{2+\alpha}{2}}(\bar{Q})} \leq r_0$$

implies

$$(2.17) \quad \|w\|_{C^{2+\alpha, \frac{2+\alpha}{2}}(\bar{Q})} \leq r_0,$$

provided  $r_0, t_0 > 0$  are adjusted sufficiently small.

2. Now (2.8) implies

$$(2.18) \quad \begin{aligned} A(R, z, x, t) = & t \frac{\partial F}{\partial r_{ij}}(D^2 g, g) h_{x_i x_j} + t \frac{\partial F}{\partial z}(D^2 g, g) h \\ & + \int_0^1 (1-s) \frac{\partial^2 F}{\partial r_{ij} \partial r_{kl}}(D^2 g + st D^2 h + sR, g + sth + sz) ds \\ & \times (th_{x_i x_j} + r_{ij})(th_{x_k x_l} + r_{kl}) \\ & + 2 \int_0^1 (1-s) \frac{\partial^2 F}{\partial r_{ij} \partial z}(D^2 g + st D^2 h + sR, g + sth + sz) ds \\ & \times (th_{x_i x_j} + r_{ij})(th + z) \\ & + \int_0^1 (1-s) \frac{\partial^2 F}{\partial z^2}(D^2 g + st D^2 h + sR, g + sth + sz) ds \\ & \times (th + z)^2. \end{aligned}$$

Recall also that

$$(2.19) \quad \|uv\|_{C^{\alpha, \frac{\alpha}{2}}(\bar{Q})} \leq C \|u\|_{C^{\alpha, \frac{\alpha}{2}}(\bar{Q})} \|v\|_{C^{\alpha, \frac{\alpha}{2}}(\bar{Q})}$$

for all  $u, v \in C^{\alpha, \frac{\alpha}{2}}(\bar{Q})$ , and

$$(2.20) \quad \|t\|_{C^{\alpha, \frac{\alpha}{2}}(\bar{Q})} \leq C t_0^{1-\frac{\alpha}{2}}.$$

Then (2.14), (2.18), (2.20) imply

$$(2.21) \quad \|B\|_{C^{\alpha, \frac{\alpha}{2}}(\bar{Q})} \leq C(r_0^2 + t_0^{1-\frac{\alpha}{2}}).$$

Similarly formula (2.9) implies

$$(2.22) \quad \|b\|_{C^{1+\alpha, \frac{1+\alpha}{2}}(\Sigma)} \leq C(r_0^2 + t_0^{\frac{1-\alpha}{2}})$$

3. Inserting estimates (2.21), (2.22) into (2.13) we discover

$$(2.23) \quad \|w\|_{C^{2+\alpha, \frac{2+\alpha}{2}}(\bar{Q})} \leq C(r_0^2 + t_0^{\frac{1-\alpha}{2}}),$$

the constant  $C$  independent of  $r_0$  and  $t_0$ . Let  $r_0 \leq \frac{1}{2C}$ . Then (2.23) gives

$$\|w\|_{C^{2+\alpha, \frac{2+\alpha}{2}}(\bar{Q})} \leq \frac{r_0}{2} + C t_0^{\frac{1-\alpha}{2}} \leq r_0$$

if  $t_0 = t_0(r_0)$  is small enough. This verifies inequality (2.17).  $\square$

Finally we check that  $T: Y \rightarrow Y$  is a strict contraction.

**Lemma 2.4.** *If  $r_0, t_0 > 0$  are small enough,*

$$(2.24) \quad \|T(\hat{w}_1) - T(\hat{w}_2)\|_{C^{2+\alpha, \frac{2+\alpha}{2}}(\bar{Q})} \leq \frac{1}{2} \|\hat{w}_1 - \hat{w}_2\|_{C^{2+\alpha, \frac{2+\alpha}{2}}(\bar{Q})}$$

for all  $\hat{w}_1, \hat{w}_2 \in Y$ .

*Proof.* 1. Choose  $\hat{w}_1, \hat{w}_2 \in Y$ ; then

$$\|\hat{w}_1, \hat{w}_2\|_{C^{2+\alpha, \frac{2+\alpha}{2}}(\bar{Q})} \leq r_0.$$

Set

$$(2.25) \quad \begin{cases} B_1(x, t) \equiv A(D^2\hat{w}_1, \hat{w}_1, x, t), & B_2(x, t) \equiv A(D^2\hat{w}_2, \hat{w}_2, x, t), \\ b_1(x, t) \equiv a(D\hat{w}_1, x, t), & b_2(x, t) \equiv a(D\hat{w}_2, x, t), \end{cases}$$

and write  $w_1 \equiv T(\hat{w}_1)$ ,  $w_2 \equiv T(\hat{w}_2)$ . Thus  $w_1$  solves the linear PDE (2.11) with  $B_1, b_1$  replacing  $B, b$ , and  $w_2$  solves (2.11) with  $B_2, b_2$  replacing  $B, b$ . According to Lemma 2.2

$$(2.26) \quad \|w_1 - w_2\|_{C^{2+\alpha, \frac{2+\alpha}{2}}(\bar{Q})} \leq C(\|B_1 - B_2\|_{C^{\alpha, \frac{\alpha}{2}}(\bar{Q})} + \|b_1 - b_2\|_{C^{1+\alpha, \frac{1+\alpha}{2}}(\Sigma)}).$$

2. Recall that

$$(2.27) \quad \begin{aligned} & \|\Phi(u) - \Phi(v)\|_{C^{\alpha, \frac{\alpha}{2}}(\bar{Q})} \\ & \leq C(\|u\|_{C^{\alpha, \frac{\alpha}{2}}(\bar{Q})} + \|v\|_{C^{\alpha, \frac{\alpha}{2}}(\bar{Q})} + 1)\|u - v\|_{C^{\alpha, \frac{\alpha}{2}}(\bar{Q})} \end{aligned}$$

if  $u, v \in C^{\alpha, \frac{\alpha}{2}}(\bar{Q})$  and  $\Phi$  is smooth, with  $D\Phi, D^2\Phi$  bounded.

Utilizing then formulas (2.18)–(2.20), (2.27) we compute

$$(2.28) \quad \|B_1 - B_2\|_{C^{\alpha, \frac{\alpha}{2}}(\bar{Q})} \leq C(r_0 + t_0^{1-\frac{\alpha}{2}})\|\hat{w}_1 - \hat{w}_2\|_{C^{2+\alpha, \frac{2+\alpha}{2}}(\bar{Q})}.$$

Similarly

$$(2.29) \quad \|b_1 - b_2\|_{C^{1+\alpha, \frac{1+\alpha}{2}}(\Sigma)} \leq C(r_0 + t_0^{\frac{1-\alpha}{2}})\|\hat{w}_1 - \hat{w}_2\|_{C^{2+\alpha, \frac{2+\alpha}{2}}(\bar{Q})}.$$

Combining (2.26), (2.28), (2.29), we obtain estimate (2.24), provided  $r_0, t_0 > 0$  are small.  $\square$

We at last apply Banach's Fixed Point Theorem to establish

**Theorem 2.5.** *If  $\delta_0, t_0 > 0$  are sufficiently small, there exists a unique solution  $v \in C^{2+\alpha, \frac{2+\alpha}{2}}(\bar{Q}) \cap C^\infty(Q)$  of the PDE (1.13).*

*In particular  $(D^2v, v) \in G$  in  $Q$ .*

### 3. MOTION OF THE ZERO LEVEL SET BY MEAN CURVATURE

This section we devote to proving that the sets

$$(3.1) \quad \Gamma_t \equiv \{x \in V \mid v(x, t) = 0\} \quad (0 \leq t \leq t_0)$$

are in fact smooth hypersurfaces evolving by mean curvature.



**Theorem 3.1.** *We have*

$$(3.2) \quad |Dv|^2 = 1 \quad \text{in } Q.$$

*Proof.* 1. Let  $w \equiv |Dv|^2 - 1 \in C^1(\overline{Q}) \cap C^\infty(Q)$ . Then

$$(3.3) \quad w = 0 \quad \text{on } \Sigma$$

and

$$(3.4) \quad w = 0 \quad \text{on } V \times \{t = 0\},$$

according to (1.11) and the PDE (1.13).

2. Differentiating (1.13) we compute

$$v_{tx_k} = \frac{\partial F}{\partial r_{ij}}(D^2v, v)v_{x_i x_j x_k} + \frac{\partial F}{\partial z}(D^2v, v)v_{x_k} \quad (k = 1, \dots, n).$$

Thus

$$(3.5) \quad \begin{aligned} w_t &= 2v_{x_k}v_{x_k t} \\ &= 2\frac{\partial F}{\partial r_{ij}}(D^2v, v)v_{x_k}v_{x_i x_j x_k} + 2\frac{\partial F}{\partial z}(D^2v, v)|Dv|^2 \\ &= \frac{\partial F}{\partial r_{ij}}(D^2v, v)w_{x_i x_j} - 2\frac{\partial F}{\partial r_{ij}}(D^2v, v)v_{x_k x_i}v_{x_k x_j} + 2\frac{\partial F}{\partial z}(D^2v, v)|Dv|^2. \end{aligned}$$

Now

$$F(D^2v, v) = f(\dots, \lambda_i(D^2v), \dots, v) \quad \text{in } Q.$$

Thus

$$(3.6) \quad \frac{\partial F}{\partial z}(D^2v, v) = \frac{\partial f}{\partial z}(\dots, \lambda_i(D^2v), \dots, v) = \sum_{i=1}^n \frac{\lambda_i(D^2v)^2}{(1 - \lambda_i(D^2v)v)^2}.$$

On the other hand (see Caffarelli, Nirenberg, and Spruck [1])

$$\begin{aligned} \frac{\partial F}{\partial r_{ij}}(D^2v)v_{x_k x_i}v_{x_k x_j} &= \sum_{i=1}^n \frac{\partial f}{\partial \lambda_i}(\dots, \lambda_i(D^2v), \dots, v)\lambda_i(D^2v)^2 \\ &= \sum_{i=1}^n \frac{\lambda_i(D^2v)^2}{(1 - \lambda_i(D^2v)v)^2} = \frac{\partial F}{\partial z}(D^2v, v) \quad \text{by (3.6)}. \end{aligned}$$

Consequently (3.5) transforms to read

$$(3.7) \quad w_t = \frac{\partial F}{\partial r_{ij}}(D^2v, v)w_{x_i x_j} + 2\frac{\partial F}{\partial z}(D^2v, v)w \quad \text{in } Q.$$

In view of Lemma 2.1 this is a uniformly parabolic equation. As (3.3), (3.4) assert  $w = 0$  on the parabolic boundary of  $Q$ , we deduce  $w = 0$  everywhere within  $Q$ .  $\square$

Owing to (3.1), (3.2) we see that

$$\Gamma \equiv \{(x, t) \in \overline{Q} \mid v = 0\}$$

is a smooth hypersurface in  $\mathbf{R}^{n+1} \cap Q$  and each slice  $\Gamma_t \equiv \{x \in V \mid (x, t) \in Q\}$  is a smooth hypersurface in  $V$ .

**Theorem 3.2.** *The surfaces  $\{\Gamma_t\}_{0 \leq t \leq t_0}$  comprise a classical motion by mean curvature starting from  $\Gamma_0$ .*

*Proof.* For each fixed  $t$  in  $[0, t_0]$ ,  $\nu \equiv Dv$  is a normal unit vector field to  $\Gamma_t$  in  $\mathbf{R}^n$ . In addition the PDE (1.13) implies

$$v_t = \Delta v = \operatorname{div}(\nu) \quad \text{on } \Gamma_t.$$

However

$$\operatorname{div}(\nu) = -(n-1)H = -(\kappa_1 + \cdots + \kappa_{n-1}),$$

where  $\kappa_1, \dots, \kappa_{n-1}$  are the principal curvatures of  $\Gamma_t$  computed with respect to  $\nu$  and

$$H = \frac{1}{n-1}(\kappa_1 + \cdots + \kappa_{n-1})$$

is the mean curvature.

Now fix  $0 \leq t \leq t_0$ ,  $x \in \Gamma_t$ , and evolve the point  $x$  according to the nonautonomous ODE

$$\begin{cases} \dot{x}(s) = -[\operatorname{div}(\nu)\nu](x(s), s) & (s > t), \\ x(t) = x. \end{cases}$$

Then

$$\begin{aligned} \frac{d}{ds}v(x(s), s) &= -[(Dv \cdot \nu)\operatorname{div}(\nu)](x(s), s) + v_t(x(s), s) \\ &= -\operatorname{div}(\nu) + v_t = 0. \end{aligned}$$

Thus

$$v(x(s), s) = 0 \quad (s > t),$$

and this implies  $\{\Gamma_t\}_{0 \leq t \leq t_0}$  is evolving by mean curvature.  $\square$

#### 4. INSTANTANEOUS TEARING APART

As a further application of the PDE (1.7)–(1.9) we prove in this section that two distinct smooth surfaces evolving by mean curvature must instantly tear completely apart, even if they initially touched on a large set.

More precisely, suppose as above  $\Gamma_0$  is the smooth connected boundary of the bounded open set  $U \subset \mathbf{R}^n$ . Assume also that  $\widehat{\Gamma}_0$  is the smooth connected boundary of a bounded open subset  $\widehat{U} \subset \mathbf{R}^n$  and that

$$(4.1) \quad \widehat{\Gamma}_0 \subset \overline{U}.$$

Thus the surface  $\widehat{\Gamma}_0$  lies within the closed region  $\overline{U}$  bounded by  $\Gamma_0$ . Suppose further that

$$(4.2) \quad \widehat{\Gamma}_0 \neq \Gamma_0.$$

Choose next  $t_0 > 0$  so small that the classical evolutions  $\{\Gamma_t\}$  and  $\{\widehat{\Gamma}_t\}$  starting at  $\Gamma_0$  and  $\widehat{\Gamma}_0$ , respectively, exist at least for times  $0 \leq t \leq t_0$ .

**Theorem 4.1.** *We have*

$$(4.3) \quad \Gamma_t \cap \widehat{\Gamma}_t = \emptyset \quad \text{for } 0 < t \leq t_0.$$

We see therefore that the two evolutions by mean curvature instantly completely “tear apart”. The point is that the PDE describing evolution by mean

curvature is “uniformly parabolic along the surface” and thus admits infinite propagation speed for disturbances. See [3, §8] for some geometric consequences.

*Proof.* 1. Fix  $\delta_0$  so small that the signed distance function

$$d(x, t) \equiv \begin{cases} -\text{dist}(x, \Gamma_t) & \text{if } x \in U_t, \\ \text{dist}(x, \Gamma_t) & \text{if } x \in \mathbf{R}^n - U_t \end{cases}$$

is smooth in

$$Q \equiv \{(x, t) \mid -\delta_0 \leq d(x, t) \leq \delta_0, 0 \leq t \leq t_0\}.$$

(Here  $U_t$  is the bounded region enveloped by  $\Gamma_t$ .) As above we have

$$(4.4) \quad d_t = F(D^2 d, d) \quad \text{in } Q.$$

Similarly the signed distance

$$\hat{d}(x, t) \equiv \begin{cases} -\text{dist}(x, \hat{\Gamma}_t) & \text{if } x \in \hat{U}_t, \\ \text{dist}(x, \hat{\Gamma}_t) & \text{if } x \in \mathbf{R}^n - \hat{U}_t \end{cases}$$

is smooth in  $Q$ , provided  $\delta_0, t_0 > 0$  are sufficiently small, and  $\hat{\Gamma}_0$  is close enough to  $\Gamma_0$ . (We denote by  $\hat{U}_t$  the region bounded by  $\hat{\Gamma}_t$ .) Also

$$(4.5) \quad \hat{d}_t = F(D^2 \hat{d}, \hat{d}) \quad \text{in } Q$$

as well.

2. Now since (4.1) holds, we have  $\hat{\Gamma}_t \subseteq \bar{U}_t$  for each  $0 \leq t \leq t_0$ . Thus  $\hat{d} \geq d$  in  $Q$ .

3. Set  $w \equiv \hat{d} - d$ . Then

$$(4.6) \quad w \geq 0 \quad \text{in } Q, \quad w \neq 0 \quad \text{on } Q \cap \{t = 0\},$$

and from (4.4), (4.5) it follows that

$$(4.7) \quad w_t = a_{ij} w_{x_i x_j} + c w \quad \text{in } Q$$

for

$$a_{ij} \equiv \int_0^1 \frac{\partial F}{\partial r_{ij}}(t D^2 \hat{d} + (1-t) D^2 d, t \hat{d} + (1-t) d) dt$$

and

$$c \equiv \int_0^1 \frac{\partial F}{\partial z}(t D^2 \hat{d} + (1-t) D^2 d, t \hat{d} + (1-t) d) dt.$$

According to Lemma 2.1 the coefficients  $((a_{ij}))$  are uniformly positive definite if  $\delta_0, t_0$  are small enough and  $\hat{\Gamma}_0$  is close enough to  $\Gamma_0$ . From (4.6), (4.7) and the strong maximum principle we deduce that

$$(4.8) \quad w > 0 \quad \text{in the interior of } Q.$$

Assertion (4.3) follows.

4. In the general case that  $\hat{\Gamma}_0$  verifies (4.1), (4.2), but is not necessarily everywhere close to  $\Gamma_0$ , we interpolate a smooth surface  $\tilde{\Gamma}_0$  between  $\Gamma_0$  and  $\hat{\Gamma}_0$ , so that  $\tilde{\Gamma}_0$  satisfies (4.1), (4.2) and is close to  $\Gamma_0$ . Then, by steps 1–3,

$$\tilde{\Gamma}_t \cap \Gamma_t = \emptyset$$

for all small  $t > 0$ ,  $\{\tilde{\Gamma}_t\}$  denoting the classical evolution from  $\tilde{\Gamma}_0$ . As  $\tilde{\Gamma}_t$  lies within the region bounded by  $\Gamma_t$  and  $\hat{\Gamma}_t$  lies in the region bounded by  $\tilde{\Gamma}_t$ , assertion (4.3) follows in this case as well.  $\square$

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